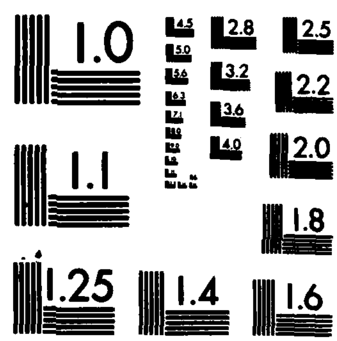
AD-A140 953 THE POSTPROCESSING TECHNIQUE IN THE FINITE ELEMENT METHOD - THE THEORY AN..(U) MARYLAND UNIV COLLEGE PARK LAB FOR NUMERICAL ANALYSIS I BABUSKA ET AL. MAR 84 1/1

UNCLASSIFIED BN-1016 N00014-77-C-0623 F/G 12/1 NL



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

INSTITUTE FOR PHYSICAL SCIENCE
AND TECHNOLOGY

(15)

AD-A140 953

Laboratory for Numerical Analysis

Technical Note BN-1016

THE POSTPROCESSING TECHNIQUE IN THE FINITE ELEMENT METHOD THE THEORY AND EXPERIENCE

by

I. Babuška
Department of Mathematics
and
Institute for Physical Science and Technology
University of Maryland
College Park, MD 20742

and

K. Izadpanah
B. Szabo

Washington University
St. Louis, MO 63130

March 1984

UNIVERSITY OF MARYLAND

Encl (1)

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER Technical Note BN-1016	2. GOVT ACCESSION NO. AD-A140 753	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) The postprocessing technique in the finite element method - The theory and experience		5. TYPE OF REPORT & PERIOD COVERED Final life of the contract
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) *I. Babuska, K. Izadpanah and B. Szabo		8. CONTRACT OR GRANT NUMBER(s) ONR N00014-77-C-0623
9. PERFORMING ORGANIZATION NAME AND ADDRESS *Institute for Physical Science and Technology University of Maryland College Park, MD 20742		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Department of the Navy Office of Naval Research Arlington, VA 22217		12. REPORT DATE March 1984
		13. NUMBER OF PAGES 25
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The paper addresses the h, p, and h-p versions of the finite element method in connection with a postprocessing technique for extracting the values of a functional. This technique combines the finite element method with the analytical ideas of the theory of partial differential equiations of elliptic type.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-LF-014-6601

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

THE POSTPROCESSING TECHNIQUE IN THE FINITE ELEMENT METHOD. THE THEORY AND EXPERIENCE

I. Babuska[1]

University of Maryland, College Park, Maryland, U.S.A.

K. Izadpanah[2]

B. Szabo[2]

Washington University, St. Louis, Missouri, U.S.A.

The paper addresses the h, p, and h-p versions of the finite element method in connection with a postprocessing technique for extracting the values of a functional. This technique combines the finite element method with the analytical ideas of the theory of partial differential equations of elliptic type.

1. INTRODUCTION

Finite element computations in structural mechanics usually have two purposes: (1) to determine the stress and displacement fields and (2) to determine the values of certain functionals defined on displacement fields as, for example, the stress intensity factors, stresses at specific points, reactions, etc. Computations of these values involve the finite element solution. For example, the stress components are often computed at the Gauss points of the elements and the stresses at any other points are then computed by the interpolation technique, the stress intensity factors is determined by the J-integral or curve fitting technique, etc. We shall refer to these operations as **postprocessing**.

Usually the values of these functionals are needed to be known with higher accuracy and reliability than the displacement or stress field itself.

[1] Partially supported by the Office of Naval Research under grant number N00014-77-C-0623.

[2] Partially supported by the Office of Naval Research under grant number N00014-81-K-0625.

Assuming that we have the finite element solution and wish to determine certain functional values the following questions arise:

1) What should the relationship be between the computational effort spent on the finite element solution and the effort spent on postprocessing: Is it better to use a very simple and inexpensive postprocessing technique as for example direct evaluation of the stresses from the finite element solution in the desired points or should one select a more expensive technique. Of course we have to relate the answer to the achieved accuracy and to the reliability and robustness of the postprocessing procedures under consideration.

2) Given a finite element solution, what is the largest accuracy of the functional values one can achieve by the postprocessing technique. In other words, what is the maximal information contained in the finite element solution which could be used for the extraction of the desired value.

3) How do the various versions of the finite element method, i.e., the h-version, the p-version and the h-p version bear on the importance of proper selection of the postprocessing techniques.

These questions are discussed in some details.

2. THE EXTENSION OPERATORS. THE h, p AND h-p VERSIONS OF THE FINITE ELEMENT METHOD

There are three versions of the finite element methods based on the common variational (energy) principle. They are characterized by the systematic selection (extension) of the finite element spaces leading to the convergence of the finite element solutions to the exact one.

The **h-version** is the classical and most commonly used method of extension: the polynomial degree of elements p is fixed and mesh refinement is used for controlling the error of approximation (h refers to the size of the element). Typically the polynomial degree of elements is low, usually $p = 1$ or 2. Proper selection of the mesh and its refinement strongly influences the error and its dependence on the computational effort.

In the **p-version** the mesh is fixed and the polynomial degree of elements is increased either uniformly or selectively over the mesh.

The **h-p version** combines the h and p-versions, i.e., error reduction is achieved by a proper mesh refinement and concurrent changes in the distribution of the polynomial degree of elements.

The performance of the various extension operators can be compared from various points of view, the most important of which are human and computer resource requirements in relation to the desired level of precision. Such relationships are difficult to quantify and are subject due to various factors, therefore the performance of the extension operators is usually related to the number of degrees of freedom N. Of course evaluation of an extension operator would not be meaningful without considering the goals of computation. For example, if only stress intensity factors are desired, then the accuracy of the computed displacements, reactions or stresses are not of importance. In many cases the computation has multiple goals.

3. THE MODEL PROBLEM

In order to illustrate the essential properties of finite element solution and extraction techniques, we have selected a model problem which represents some of key features of a large class of engineering problems. Specifically let us consider the plane strain problem of two-dimensional elasticity (homogeneous isotropic material) with E and ν representing the modulus of elasticity and Poisson ratio respectively ($E > 0$, $0 < \nu < .5$). The domain D, a square panel with a crack, is shown in Fig. 1.

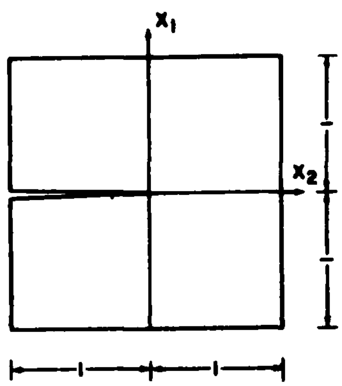


Figure 1
The model problem

We shall be concerned here with problems in which only tractions are prescribed at the boundary (i.e., first boundary value problem of elasticity).

We denote the displacement vector function by $\underline{u} = \{u_1, u_2\}$ and the corresponding stress tensor by

$$T = \begin{matrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{matrix} , \qquad \tau_{12} = \tau_{21} .$$

The strain energy functional is

$$W(u) = \frac{E}{2(1-2\nu)(1-\nu)} \int_D \Big[(1-\nu)\Big(\frac{\partial u_1}{\partial x_1}\Big)^2 + 2\nu \frac{\partial u_1}{\partial x_1}\frac{\partial u_2}{\partial x_2}$$

$$+ (1-\nu)\Big(\frac{\partial u_2}{\partial x_2}\Big)^2 + \frac{1-2\nu}{2}\Big(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}\Big)^2\Big] dx_1 dx_2 . \qquad (3.1)$$

The solution u satisfies the Navier-Lamé equations. It is possible to express the solution through two holomorphic functions $\phi(z)$, $\psi(z)$ using the theory of Muskhelishvili [1].

$$2\mu(u_1 + iu_2) = \kappa\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)} \tag{3.2}$$

where

$$z = x_1 + ix_2, \quad \mu = \frac{E}{2(1+\nu)}, \quad \kappa = 3 - 4\nu \tag{3.3}$$

and $\bar{z} = x_1 - ix_2$, resp. $\overline{\phi'(z)}$ mean conjugate values to z and $\phi'(z)$.

The components of the stress tensor are expressed by Kolosov-Muskhelishvili formulae

$$\begin{aligned} \tau_{11} + \tau_{22} &= 2(\phi'(z) + \overline{\phi'(z)}) = 4 \operatorname{Re} \phi'(z) \\ &= 2(\Phi(z) + \overline{\Phi(z)}) \end{aligned} \tag{3.4}$$

$$\tau_{22} - \tau_{11} + 2i\tau_{12} = 2[\bar{z}\phi''(z) + \psi'(z)] = 2[\bar{z}\Phi'(z) + \Psi(z)] \tag{3.5}$$

where

$$\Phi(z) = \phi'(z), \qquad \Psi(z) = \psi'(z) \tag{3.6}$$

and $\operatorname{Re} \phi'(z)$ is the real part of $\phi'(z)$.

The correspondence between the displacements (and the stress) field and the functions ϕ and ψ is one to one up to the constants γ and γ' in ϕ and ψ, respectively, satisfying the relation $\gamma - \bar{\gamma}' = 0$.

In our model problem we consider the following (exact) solution

$$\Phi(z) = (1+i)z^{-1/2} \tag{3.7}$$

$$\Omega(z) = \Phi(z) \tag{3.8}$$

$$\Omega(z) = \bar{\Phi}(z) + z\bar{\Phi}'(z) + \bar{\Psi}(z) \tag{3.9}$$

where $\bar{\Phi}(z) = \overline{\Phi(\bar{z})}$, $\bar{\Phi}'(z) = \overline{\Phi'(\bar{z})}$, $\bar{\Psi}(z) = \overline{\Psi(\bar{z})}$.

$\Omega(z)$ is a holomorphic function on D. Function $z^{-1/2}$ is to be understood as the principal branch of $z^{-1/2}$ on D. Function $\Psi(z)$ is uniquely defined by (3.9) and (3.7) (3.8). The tractions on the boundary of D are defined by (3.4) (3.5). It can be readily verified that the two edges of the crack are traction free.

We will now discuss the finite element solution and the postprocessing technique if the tractions are prescribed on the boundary of D so that the exact solution to the problem is given by (3.7)-(3.9). Specificaly we now consider the case $E = 1$, $\nu = 3$. The strain energy of the exact solution is: $W = 42.16491240$.

4. THE FINITE ELEMENT SOLUTION

We have solved the model problem by the h and p-versions of the finite element method. The p-version of the finite

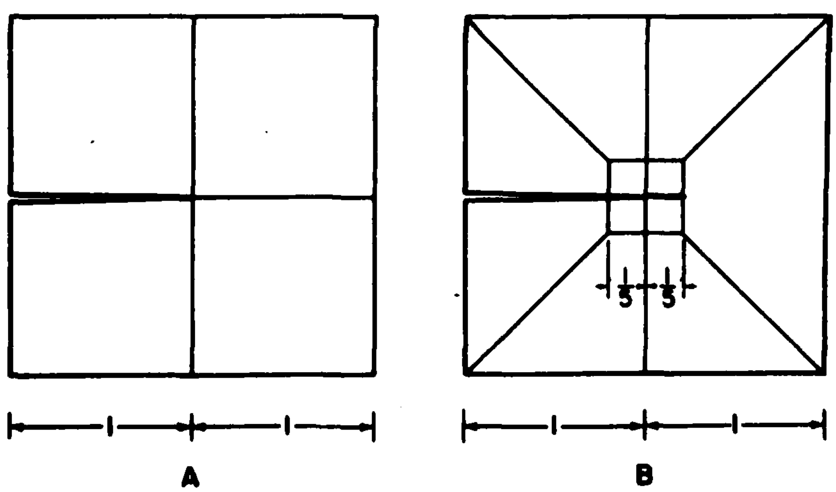


Figure 2
The meshes for the p-version, A: Mesh 1, B: Mesh 2

element method was implemented in the experimental computer program COMET-X developed at the Center for Computational Mechanics of Washington University in St. Louis [2]. The two meshes shown in Fig. 2A,B were used. The polynomial degrees were the same for all elements and ranged from 1 to 8. The shape functions on trapezoidal elements of mesh 2 were constructed by blending function technique.

The h-version solution was obtained by means of the computer program FEARS developed at the University of Maryland [3, 4].

FEARS uses quadrilateral elements of degree one. The program is adaptive and produces a sequence of nearly optimal meshes. See [3] [4] [5] [6] [7]. The mesh from this sequence with 319 elements and number of degrees of freedom $N = 617$ is shown in Fig. 3.

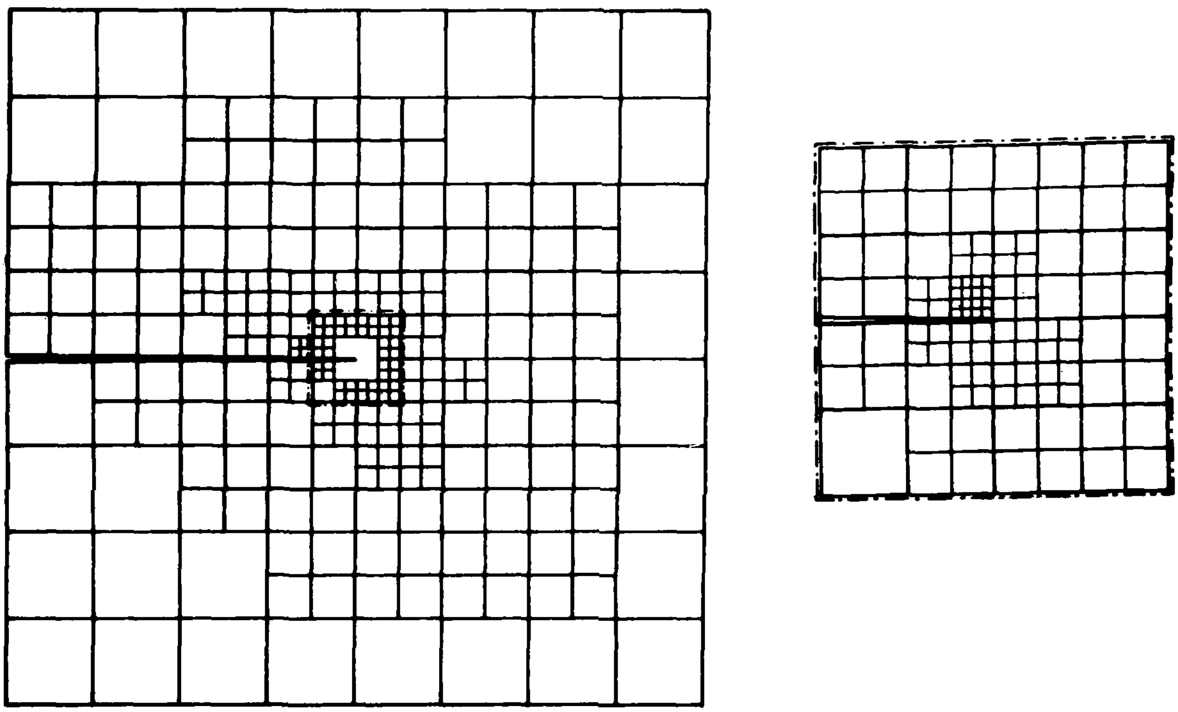


Figure 3
The mesh constructed by the adaptive program FEARS

5. ERROR OF THE FINITE ELEMENT SOLUTION MEASURED IN ENERGY NORM

We denote the exact solution by $\underline{u}_0$ and the finite element solution by $\underline{u}_{FE}$. The error of the finite element solution is denoted by $\underline{e}$,

$$\underline{e} = \underline{u}_0 - \underline{u}_{FE}.$$

We measure the magnitude of the error by the **energy norm** $\|\cdot\|_E$,

$$\|\underline{e}\|_E = [W(\underline{e})]^{1/2}. \qquad (5.1)$$

This measure is equivalent to measuring the error in the stress components by integrals of their squares (the L_2 norm). In our case when tractions are specified at the boundary

$$W(\underline{u}_{FE}) < W(\underline{u}_0) \tag{5.2}$$

and

$$\|e\|_E = [W(\underline{u}_0) - W(\underline{u}_{FE})]^{1/2} . \tag{5.3}$$

The extension operators under consideration monotonically increase the finite element spaces either by increasing the degree of elements or refining the mesh. Therefore the energy norm of the error monotonically decreases. We can write

$$\|e\|_E \leq C(N)N^{-\mu} \tag{5.4}$$

and expect that for properly chosen μ the function $C(N)$ is nearly constant especially for larger N. The number $\mu > 0$ is the rate of convergence of the error measured in the energy norm.

It is possible to estimate the value of μ. In our case the rate μ is governed by the strength of the singularity of the solution. It can be shown that for the p-version [8],[9]

$$\|e\|_E < C(\varepsilon)N^{-(1/2 - \varepsilon)} \tag{5.5}$$

with $\varepsilon > 0$ arbitrarily small and C independent of N. The h-version using the uniform mesh yields the estimate

$$\|e\|_E < CN^{-1/4} \tag{5.6}$$

with the rate independent of the degree of elements. The optimal refinement of the mesh leads to the estimate

$$\|e\|_E < CN^{-p/2} \tag{5.7}$$

(FEARS uses $p = 1$) where the rate is independent of the strength of the singularity.

The h-p version with optimal mesh and p-distribution leads to the estimate

$$\|\underline{e}\|_E < Ce^{-\gamma N^{\theta}}$$

where $\theta = 1/3$ independently of the strength of the singularity and $\gamma > 0$.

The relative error in the energy norm defined as

$$\|\underline{e}\|_{E,R} = \frac{\|\underline{e}\|_E}{\|\underline{u}_0\|_E} \tag{5.8}$$

has been plotted in Fig. 4 on log-log scale for the p-version (mesh 1,2), for the h-version with

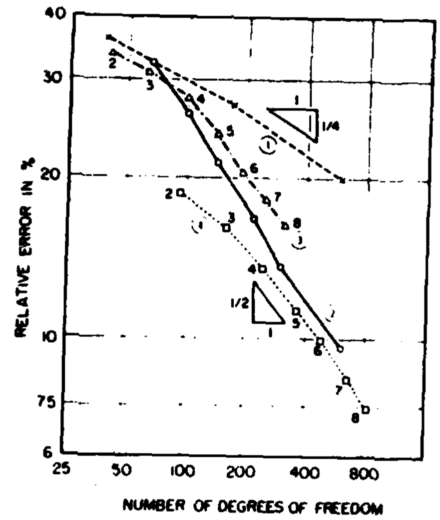


Figure 4
Relative error in the energy norm vs degrees of freedom
(1) h-version, uniform mesh, (2) h-version, adaptively constructed mesh, (3) p-version Mesh 1, (4) p-version Mesh 2

adaptively constructed mesh and for the h-version with uniform mesh. The polynomial degree of elements is also shown in the figure. The shown slopes are the theoretical slopes of the rate of convergence [$\mu = 1/2$ and $1/4$]. It is seen that the observed rate of convergence closely agrees with (5.5)(5.7). From (5.4) we can compute $C(N)$ for the p-version. The results are given in Table 1.

Tables 2 and 3 show analogous results for the h-version. The comparison between Tables 1-3 shows that for 5% accuracy we need $N = 1770$ when using p-version Mesh 2, $N = 2290$ for h-version with adaptively refined mesh and $N = 146000$ for h-version with uniform mesh.

TABLE 1

Relationship between $\|e\|_{E,R}$ and N for the p-version, Mesh 2 [$\mu = 1/2$]

p	N	$\|e\|_{E,R}$	$C(N)/\|u_0\|_E$
1	35	32.61%	2.010
2	95	18.35%	1.816
3	135	15.89%	1.997
4	239	13.24%	2.059
5	347	11.06%	2.061
6	479	9.47%	2.079
7	635	8.27%	2.088
8	815	7.37%	2.099

TABLE 2

Relationship between $\|e\|_{E,R}$ and N for the h-version with adaptively constructed mesh [$\mu = 1/2$]

N	$\|e\|_{E,R}$	$C(N)/\|u_0\|_E$
67	32.91%	2.035
101	26.38%	2.665
143	21.35%	2.562
221	16.79%	2.501
301	13.61%	2.366
617	9.63%	2.394

TABLE 3

Relationship between $\|e\|_{E,R}$ and N for the h-version with uniform mesh [$\mu = 1/4$]

N	$\|e\|_{E,R}$	$C(N)/\|u_0\|_E$
51	36.02%	.967
167	27.07%	.974
591	19.81%	.977

6. COMPUTATION OF THE STRESSES

The finite element method provides the solution u_{FE} which converges to the exact solution in the energy norm. We have seen that the error measured in this norm decreases monotonically and in a very orderly way. We now examine the pointwise error an stresses for the h and p-versions. We denote the error in the stress components as

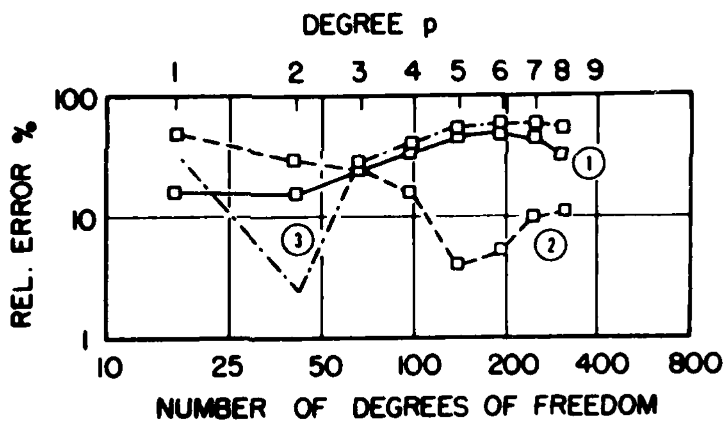
$$e_{ij}(x_1,x_2) = \tau_{ij}^{[0]}(x_1,x_2) - \tau_{ij}^{[FE]}(x_1,x_2) \qquad (6.1)$$

and the relative error by

$$e_{ij}^{R}(x_1,x_2) = \frac{|e_{ij}(x_1,x_2)|}{|\tau_{ij}^{[0]}(x_1,x_2)|}$$

where $\tau_{ij}^{[0]}$ and $\tau_{ij}^{[FE]}$ are respectively the stress components corresponding to the exact and finite element solutions. We will compute the stresses directly from the derivatives of $\bar{u}_{FE}$ and the stress-strain law. Fig. 5 shows the relative error e_{ij}^{R} in τ_{ij} at the point (.0,.1) computed by the p-version.

a



b

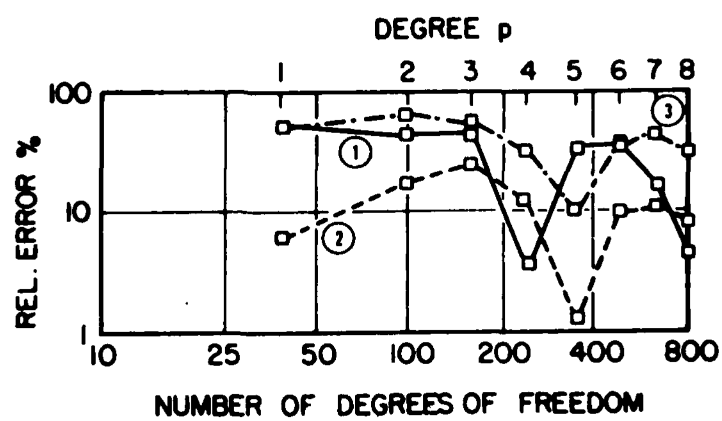


Figure 5
The relative error of e_{ij}^{R} computed by the p-version
a) Mesh 1, b) Mesh 2. (1) e_{11}^{R}, (2) e_{22}^{R}, (3) e_{12}^{R}

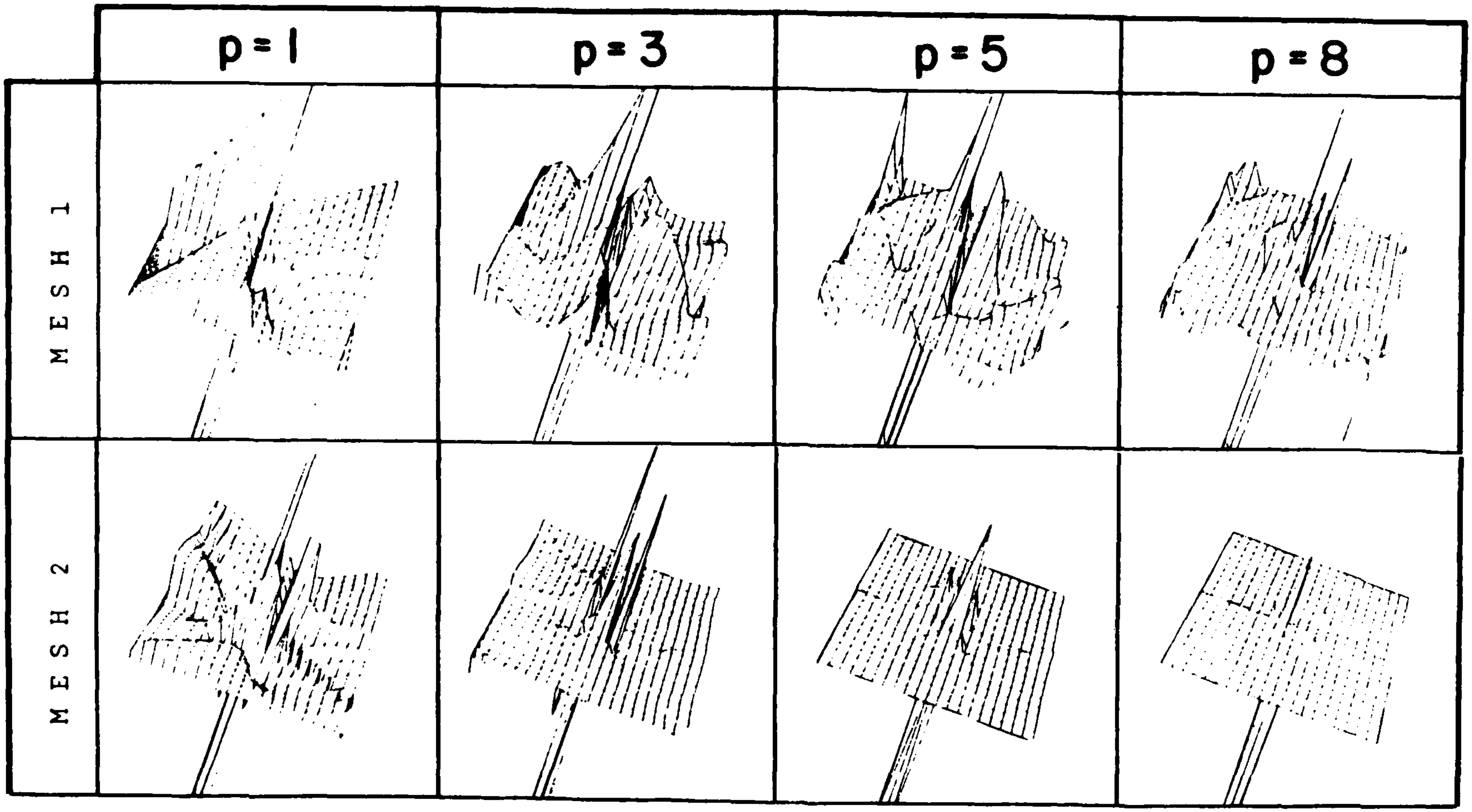


Figure 6

Fig. 6 shows isometric drawings of the error in τ_{22} for various p values for meshes 1 and 2. The error values were computed on a uniform grid with the grid points (ih, jh) $h = .1$, $i, j = -10, 10$. At points other than the grid points, the values were computed by linear interpolation.

In the case of the h-version, the error is discontinuous at the boundary of every element. Therefore we compute the stresses in the center of every element where increased accuracy can be expected.

In Fig. 7 we show the level-lines of the error in τ_{22} (using the mesh shown in Fig. 3) in the upper right quarter of the domain D. The local maxima and minima are shown also in the figure. The error is large in the neighborhood of the tip of the crack. The level-lines and the local maxima and minima depend on the interpolation technique used. We see in contrast to the p-version that the oscillatory behaviour of the error is not so strong here; nevertheless, it has to be emphasized that when the stresses are computed everywhere directly from the displacements, strong oscillatory behaviour will appear in every element.

The center of the elements are changing with the mesh. To show the convergence of the stresses, we selected for the Table 4 the center points which are closest to the tip of the crack (in the first quarter of D). The table shows the error in % and the magnitude of the exact values of the stress.

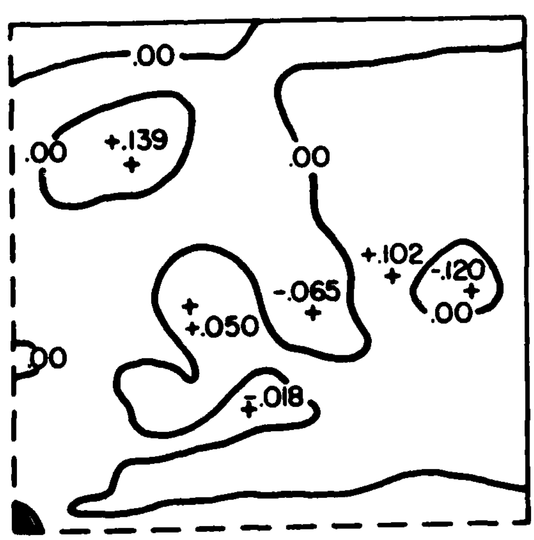


Figure 7
The level lines of the error e_{22} of τ_{22} in the upper quarter of D computed by the h-version

TABLE 4

The relative error of the stresses in the neighborhood of the origin.

No. of elements	N	Coordinates		$\lvert e^R_{11}\rvert$ $\lvert\tau^{[0]}_{11}\rvert$	$\lvert e^R_{22}\rvert$ $\lvert\tau^{[0]}_{22}\rvert$	$\lvert e^E_{12}\rvert$ $\lvert\tau^{[0]}_{12}\rvert$	
		x_1	x_2				
16	51	.25	.25	35.54% 5.037	19.90% 3.751	20.70% 1.553	Adaptive Mesh
43	143	.125	.125	33.95% 7.124	10.84% 5.304	15.14% 2.197	Adaptive Mesh
106	221	.3125(-1)	.3125(-1)	31.01% 14.251	6.09% 10.609	12.35% 4.394	Adaptive Mesh
319	617	.7125(-2)	.7125(-2)	29.77% 28.501	2.40% 21.218	17.65% 8.789	Adaptive Mesh
16	51	.25	.25	35.54% 5.037	19.90% 3.751	20.70% 1.553	Uniform Mesh
64	167	.125	.125	36.34% 7.124	20.07% 5.304	14.52% 2.197	Uniform Mesh
256	591	.625(-2)	.625(-1)	34.46% 10.076	17.16% 7.502	14.09% 3.107	Uniform Mesh

To depict the behaviour in a fixed point (.25, .25) we select the center points closest to it. Table 5 shows the results. If we desire to compute the stress components in the nodal point (.25, .25) we have 4 values for disposition and also their average. Table 6 we shows the relative errors. The value in the lines 1, 2, 3, 4 are computed from the elements ordered counterclockwise starting with the upper-right one. The line A shows relative error of the average of the stress values computed in the four elements.

In contrast to the monotonic and orderly behaviour of the error measured in the energy norm, the accuracy in the stresses is poor and nonmonotonic, although the stresses are converging in integral sense (in the energy norm) monotonically. In addition, the quality of the computed stress components is very different.

TABLE 5

The relative error of the stresses in the neighborhood of (.25,.25).

No. of elements	N	Coordinates x_1	Coordinates x_2	$\lvert e^R_{11}\rvert$ / $\lvert\tau^{[0]}_{11}\rvert$	$\lvert e^R_{22}\rvert$ / $\lvert\tau^{[0]}_{22}\rvert$	$\lvert e^R_{12}\rvert$ / $\lvert e^R_{12}\rvert$	
16	51	.25	.25	35.54% 5.037	19.90% 3.751	20.70% 1.553	Adaptive Mesh
43	143	.375	.375	1.09% 4.113	2.10% 3.062	43.51% 1.268	
106	221	.1875	.1875	3.89% 5.817	8.26% 4.331	40.28% 1.794	
319	617	.21875	.21875	.464% 5.386	.812% 4.010	12.12% 1.661	
16	51	.25	.25	35.54% 5.037	19.90% 3.751	20.70% 1.553	Uniform Mesh
64	167	.373	.375	8.01% 4.113	7.79% 3.062	16.06% 1.268	
256	591	.1875	.1875	10.20% 5.817	10.78% 4.331	6.45% 1.794	

7. POSTPROCESSING

We have seen that stresses computed directly from finite element solutions are not accurate. Nevertheless, often the values of the stresses is the main aim of the computation.

We will show now that by utilizing the analytical structure of the Navier-Lamé equations it is possible to compute stresses with the accuracy comparable to the accuracy of the energy of the finite element solution (which is the square of the error measured in the energy norm). We will outline the main idea. For more, see [8], [9], [10].

Let $x_0 = (x_{0,1}, x_{0,2}) \in D$ and denote by $S(x_0,\rho)$ the disc of radius ρ centered in $\underline{x}_0$. Further, let $D(x_0,\rho) = D - S(x_0,\rho)$. See Fig. 8. The boundary of $D(x_0,\rho)$ is denoted by $\partial D(x_0,\rho) = \partial\Omega \cup \Gamma$ where Γ is the boundary of the disk $S(x_0,\rho)$. We now define the **extraction** (displacement) function $\underline{w}(x_0,x) \equiv (w_1,w_2)$ which corresponds to the functions $\hat{\phi}$, $\hat{\psi}$ in the sense of (3.2) (3.3) and are defined as follows

$$\hat{\phi}(z) = A(z - z_0)^{-1} + \hat{\phi}_*(z) \tag{7.1}$$

TABLE 6

The relative error of the stresses in (.25.25)

No. of elements	N		e^R_{11} %	e^R_{22} %	e^R_{12} %	
43	143	A	.034	1.60	33.95	Adaptive Mesh
		1	.042	4.97	4.35	
		2	2.41	6.60	3.38	
		3	.026	1.76	72.22	
		4	3.09	3.14	64.41	
106	221	A	10.99	7.57	11.16	
		1	4.79	2.31	35.86	
		2	12.21	2.01	.093	
		3	17.27	17.42	106.43	
		4	9.71	13.01	71.49	
319	617	A	4.09	5.47	13.42	
		1	1.46	2.68	22.96	
		2	4.41	.099	55.69	
		3	4.41	13.43	3.88	
		4	6.07	11.74	28.85	
64	167	A	12.12	10.65	17.36	Uniform Mesh
		1	19.37	15.35	13.36	
		2	6.75	8.68	5.47	
		3	5.87	5.94	68.41	
		4	17.51	12.63	60.22	
256	591	A	8.11	10.13	12.66	
		1	8.10	6.62	13.38	
		2	5.38	5.05	8.15	
		3	8.12	13.64	38.71	
		4	10.84	15.84	33.48	

$$\hat{\xi}(z) = B(z-z_0)^{-1} + \hat{\xi}_*(z) \tag{7.2}$$

$$\hat{\psi}(z) = \hat{\xi}(z) - \bar{z}_0\hat{\phi}'(z) \tag{7.3}$$

where $\hat{\phi}_*(z)$ and $\hat{\xi}_*(z)$ are arbitrary holomorphic functions on D (not only on $D(x_0,\rho)$). Note that $\hat{\phi}$, $\hat{\psi}$ are holomorphic on $\Omega(x_0,\rho)$ for any $0 < \rho$.

Although the domain $D(x_0,\rho)$ is doubly connected, the displacement function w defined by (7.1) through (7.3) by (3.2) is a single valued function and it is an admissible displacement function.

Denote by $T^{[u]}$, $T^{[w]}$ the stress tensors associated with the

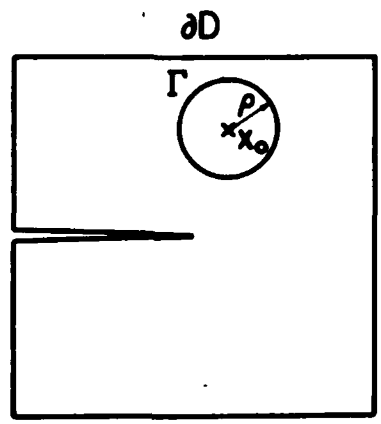


Fig. 8. The domain $D(x_0, \rho)$.

displacement functions u and w. Denote the outward normal to $\partial\Omega(x_0, \rho)$ by n. Then Betti's law can be written in the form

$$\int_{\partial D(x_0,\rho)} (\underline{u}, T^{[w]}\cdot n)ds = \int_{\partial D(x_0,\rho)} (\underline{w}, T^{[u]}\cdot n)ds \; . \tag{7.4}$$

This equation can be rewritten

$$\begin{aligned} &\int_{\partial D} [(\underline{u}, T^{[w]}\cdot n) - (\underline{w}, T^{[u]}\cdot n)]ds \\ &= \int_{\Gamma} [-(\underline{u}, T^{[w]}\cdot n) + (\underline{w}, T^{[u]}\cdot n)]ds \; . \end{aligned} \tag{7.5}$$

The functions ϕ, ψ associated to the solution $\underline{u}$ can be written in the neighborhood of z_0:

$$\phi(z) = a_0 + a_1(z-z_0) + O((z-z_0)^2) \tag{7.6}$$

$$\xi(z) = b_0 + b_1(z-z_0) + O((z-z_0)^2) \tag{7.7}$$

$$\psi(z) = \xi(z) - \bar{z}_0\phi'(z). \tag{7.8}$$

Using (7.1)-(7.3) and (7.6)-(7.8) in (7.5) and letting $\rho \to 0$ we get

$$\int_{\partial D} (u, T^{[w]}\cdot n)ds - \int_{\partial D} (w, T^{[u]}\cdot n)ds$$

$$= \frac{\Pi}{2\mu}\left[b_1(\kappa A + A) + \bar{b}_1(\kappa\bar{A} + \bar{A}) + a_1(1+\kappa)B + \bar{a}_1(1+\kappa)\bar{B}\right]. \tag{7.9}$$

By application of (3.4)-(3.6) we get

$$\tau_{11}(\underline{x}_0) = \mathrm{Re}(a_1 + \bar{a}_1 - b_1) \tag{7.10}$$

$$\tau_{22}(\underline{x}_0) = \mathrm{Re}(a_1 + \bar{a}_1 + b_1) \tag{7.11}$$

$$\tau_{12}(\underline{x}_0) = \mathrm{Im}\ b_1 . \tag{7.12}$$

By proper selection of A, B we can obtain that the right hand side of (7.9) be $\tau_{i,j}$. Note that any choice of $\hat{\phi}_*$ and $\hat{\xi}_*$ in (7.1) and (7.2) does not change the right hand side of (7.9).

In our problem when the tractions are prescribed at ∂D, the function $\underline{g}(x) = T^{[u]}\cdot n$ is given. (7.9) can therefore be written in the form

$$F = \int_{\partial D} (\underline{u}_0, T^{[w]}\cdot n)ds - \int_{\partial D} (\underline{w}, \underline{g})ds \tag{7.13}$$

where F is (for proper choice of A, B) the exact value of the stress component at $x = x_0$. Of course u_0 is not known but u_{FE} is. Therefore we define

$$F_{FE} = \int_{\partial D} (\underline{u}_{FE}, T^{[w]}\cdot n)ds - \int_{\partial D} (\underline{w}, \underline{g})ds \tag{7.14}$$

By subtracting (7.13) (7.14), the error in the extracted functional F_{FE} (provided that integrals are evaluated exactly) is

$$F - F_{FE} = \int_{\partial D} (\underline{u}_0 - \underline{u}_{FE}, T^{[w]}\cdot n)ds. \tag{7.15}$$

Let us analyze now (7.15). To this end, let $\underline{v} = (v_1, v_2)$ be the (exact) solution of the problem when tractions $T^{[w]}.n$ are prescribed at ∂D. $\underline{v} \neq \underline{w}$ because $\underline{w}$ is singular at $x = x_0$, but $\underline{v}$ is not. Existence of v is guaranteed because $T^{[w]}.n$ satisfy the equilibrium condition. We can write

$$\int_{\partial D} (\underline{u}_0 - \underline{u}_{FE}), T^{w}.n)ds = 2W(\underline{u}_0 - \underline{u}_{FE}, v) \tag{7.16}$$

where $W(u,v)$ is the usual energy scalar product associated with $W(u)$ defined in (3.1). Using one of the basic property of the finite element method, namely

$$W(\underline{u} - \underline{u}_{FE}, \underline{v}_{FE}) = 0, \tag{7.17}$$

we obtain from (7.15) (7.16)

$$F - F_{FE} = 2W(\underline{u}_0 - \underline{u}_{FE}, \underline{v} - \underline{v}_{FE})$$

and hence

$$|F - F_{FE}| \leq 2\|\underline{u}_0 - \underline{u}_{FE}\|_E \|\underline{v} - \underline{v}_{FE}\|_E . \tag{7.18}$$

So far we did not discuss the choice of $\hat{\phi}_*(z)$ and $\hat{\xi}_*(z)$. (7.18) shows that $\hat{\phi}_*$ and $\hat{\xi}_*$ should be selected so that $\|\underline{v} - \underline{v}_{FE}\|$ is at least of the order of $\|u_0 - u_{FE}\|$.

If $\|\underline{v} - \underline{v}_{FE}\|_E \approx C\|\underline{u} - \underline{u}_{FE}\|_E$ we get $|F - F_{FE}| \leq C\|u_0 - u_{FE}\|_E^2 \leq C(W(u_0) - W(u_{FE}))$ and the rate of convergence is twice that of the rate of the error measured in the energy norm. Note that inequality (7.18) is upper bound which neglects possible cancellation in the energy integral.

8. SELECTION OF THE EXTRACTION FUNCTION

When x_0 is not close to the boundary of D, then we can select $\hat{\phi}_* = \hat{\xi}_* = 0$. When x_0 is close to ∂D, then $\hat{\phi}_*$ and $\hat{\xi}_*$ should be selected so that $T^{[w]}.n = 0$ on that part of the boundary which is close to x_0. Otherwise, we would not achieve that $\|\underline{v} - \underline{v}_{FE}\|_E$ will be small.

In the following we outline briefly the procedure for constructing $\hat{\phi}_*$ and $\hat{\xi}_*$ so that $T^{[w]}.n = 0$ on the crack surfaces. To simplify the notation we will write ϕ instead of $\hat{\phi}$, etc.

Define an auxiliary function $\Omega(z)$ on D

$$\Omega(z) = \bar{\Phi}(z) + z\bar{\Phi}'(z) + \bar{\Psi}(z) \qquad (8.1)$$

Using (3.4) and (3.5) the tractions on the crack surface can be written as follows

$$\tau_{22}(z_+) - i\tau_{12}(z_+) = \Phi(z_+) + \Omega(z_-) \qquad (8.2a)$$

$$\tau_{22}(z_-) - i\tau_{12}(z_-) = \Phi(z_-) + \Omega(z_+) \qquad (8.2b)$$

where z_+ and z_- respectively denote the upper and lower surface of the crack. Using (7.1)-(7.3) we get

$$\Omega(z) = -\bar{A}(z-\bar{z}_0)^{-2}+2\bar{A}(z-z_0)(z-\bar{z}_0)^{-3}-\bar{B}(z-\bar{z}_0)^{-2}+\Omega_*(z) \qquad (8.3)$$

where

$$\Omega_*(z) = \bar{\Phi}_*(z) + z\bar{\Phi}_*'(z) + \bar{\Psi}_*(z). \qquad (8.4)$$

Setting

$$\Omega_*(z) = \Phi_*(z) \qquad (8.5)$$

we obtain

$$\tau_{22}(z_+) - i\tau_{12}(z_+) = Q(z_+) + \Phi_*(z_+) + \Phi_*(z_-) \qquad (8.6a)$$

where

$$Q(z) = -A(z-z_0)^{-2} + \bar{A}(z-\bar{z}_0)^{-2} + 2\bar{A}(z - z_0)(z - \bar{z}_0)^{-3} - \bar{B}(z - \bar{z}_0)^{-2}. \qquad (8.7)$$

Note that $Q(z_+) = Q(z_-)$. Similarly

$$\tau_{22}(z_-) - i\tau_{12}(z_-) = Q(z_+) + \Phi_*(z_-) + \Phi_*(z_+). \qquad (8.6b)$$

Now we select Φ_* so that

$$\Phi_*(z_+) + \Phi_*(z_-) = -Q(z_+). \qquad (8.8)$$

(8.5) and (8.8) define now Φ_* and Ψ_*. By this selection we achieve that $\tau_{22}(z_+) = \tau_{12}(z_+) = \tau_{22}(z_-) = \tau_{12}(z_+) = 0$. The relation (8.8) can be easily achieved. For example, for

$$\Phi = - \frac{z^{-1/2} z_0^{-1/2}}{4(z^{1/2} + z_0^{1/2})^2}$$

we get

$$\Phi(z_+) + \Phi(z_-) = \frac{1}{(-z_+ + z_0)^2}$$

which is one term in (8.7). Consequently we get the other terms and combining them (8.8) is achieved.

9. NUMERICAL PERFORMANCE OF THE EXTRACTION TECHNIQUE

We now present the results of computational experiments based on our model problem and the extraction function described in Section 8 (using $\hat{\phi}*$, $\hat{\xi}*$).

Fig. 9. shows the results analogous to those shown in Fig. 5 but stress components τ_{ij} was computed by the extraction technique. The slope shown in the figure shows the rate $\mu = 1$ (*i.e.*, the rate of the convergence of the energy and not the energy norm). For comparison the error e_{12}^R for mesh 2 computed directly (see Fig. 5b) is shown also in Fig. 9. Fig. 10 shows the isometric drawings (in the same scale as in

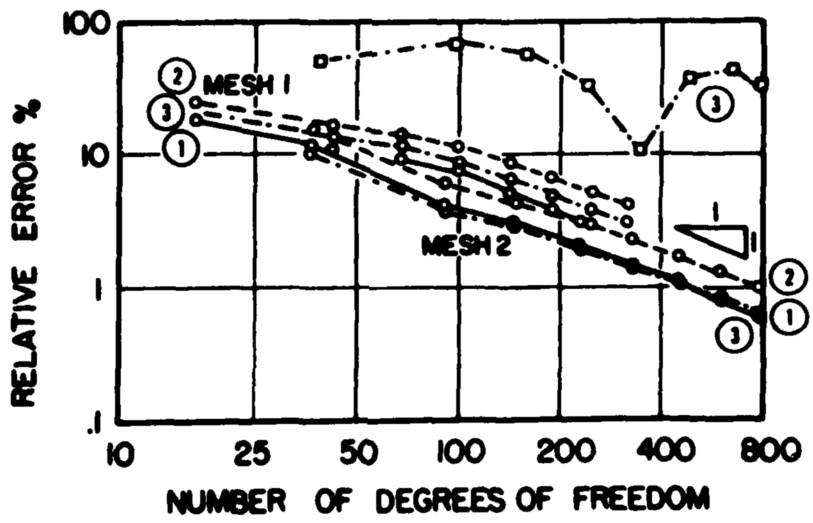


Figure 9
The relative error of τ_{ij} computed by postprocessing of the p-version for Mesh 1 and Mesh 2. (1) e_{11}^R, (2) e_{22}^R, (3) e_{12}^R

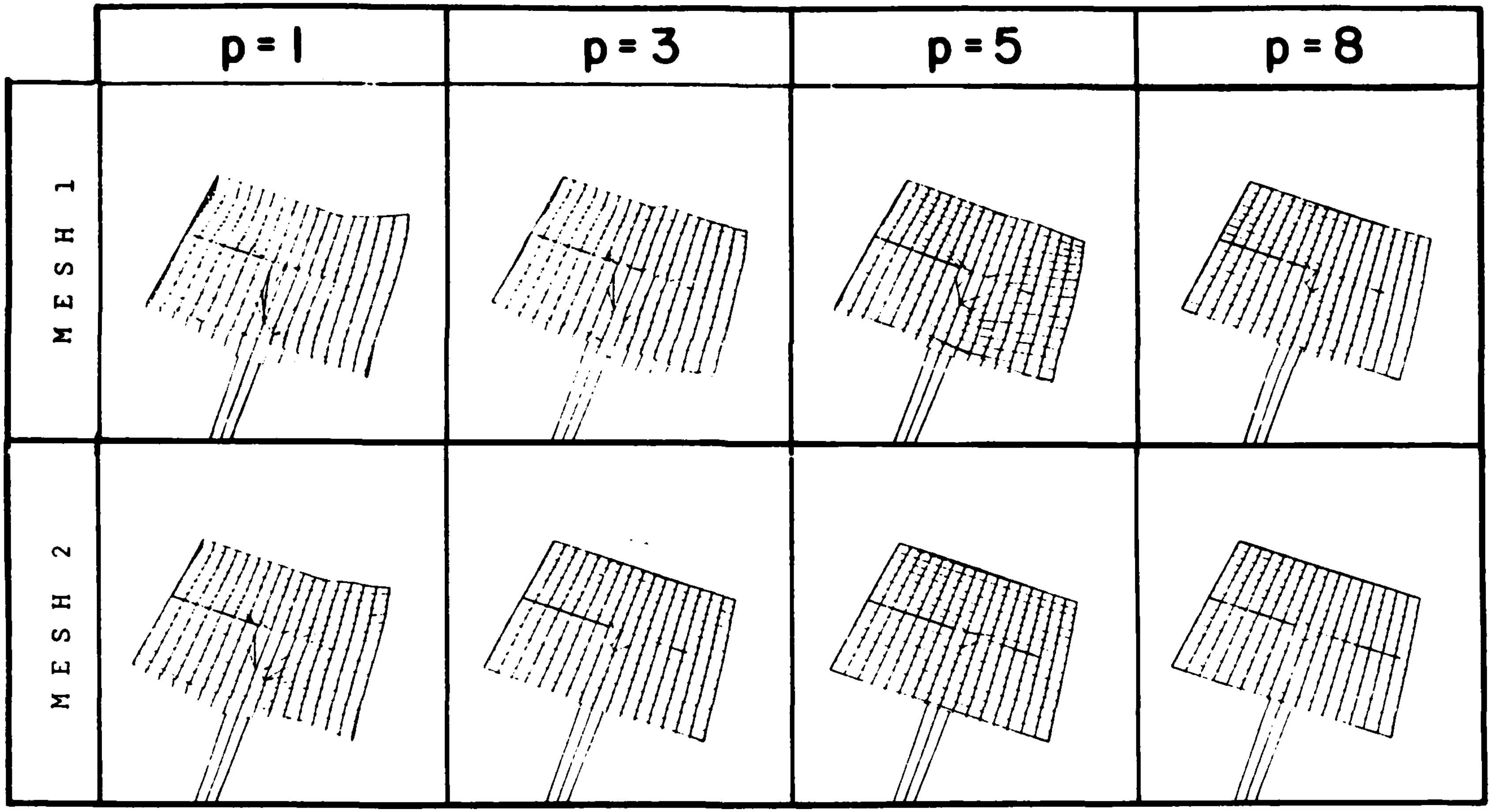


Figure 10
The behavior of the error e_{22} of τ_{22} computed by postprocessing of the p-version.

Fig. 6) of the error in τ_{22} computed by the postprocessing technique.

Table 7 shows the relative error e^R_{ij} in the stresses τ_{ij} at the point (.25,.25) computed by the postprocessing technique taking $\hat{\phi}* = \hat{\xi}* = 0$ (because the point is not close to the boundary). This data should be compared with the results of Table 5 and 6.

TABLE 7

The relative error e^R_{ij} in the stresses τ_{ij} at the point (.25,.25) computed by postprocessing.

No. of elements	N	e^R_{11}	e^R_{22}	e^R_{12}	
16	51	20.91%	22.51%	12.70%	Adaptive Mesh
43	143	11.40%	13.07%	8.29%	
106	221	4.60%	5.92%	3.74%	
319	617	1.47%	2.01%	1.31%	
16	51	20.91%	22.51%	12.70%	Uniform Mesh
64	167	12.51%	14.88%	9.86%	
256	591	6.87%	8.90%	6.28%	

We see that for adaptive meshes the error is of order N^{-1} and for uniform meshes of order $N^{-1/2}$. Similarly, as in the case of the p-version we see an orderly convergence with the rate as the square of the error measured in the energy norm (as theoretically expected).

10. CONCLUSIONS

The shown computations are characteristic in the following way. The convergence in the energy norm is monotonic and very orderly. For the smooth solution the p-version is especially effective. For unsmooth solutions the refinement of the meshes in the h-version is very essential.

The convergence of stresses in a fixed point is very "chaotic," the accuracy in various components can be very different. The rate of convergence of the postprocessed values are as the square of the error measured in the energy norm. In the case of the h-version, uniform (or piecewise uniform) meshes and smooth solution the superconvergence occurs in the center of the elements. The rate is $h^2 \log h$, i.e., effectively as the square of the error in energy norm (h). Therefore, the gain for the elements of degree 1 is not in the rate of convergence of the postprocessed value but is in the magnitude. (For $p > 1$ the gain of the postprocessing appears also in the rate.)

The postprocessing is especially important for the p-version, although it is also essential for the h-version especially for unsmooth solutions and for general meshes.

11. EFFECTIVITY OF THE POSTPROCESSING TECHNIQUE

In the introduction we raised a number of questions concerning the postprocessing. We now briefly address these question in the light of our results. Detailed analysis will be made in a forthcoming paper.

1) It is cost effective not to save computational effort on a postprocessing procedure especially when not an excessive number of extractions is made. The cost of obtaining reliable and accurate values by postprocessing is much smaller than to obtain comparable accuracy by increasing p in the p-version or refine the meshes in the h-version. The postprocessing usually removes very reliably the "chaotic" behaviour of the errors in stresses. The effectivity of the postprocessing is characterized by higher rate of convergence than in the energy norm.

2) The rate of convergence as the square of the rate of the error in the energy norm is theoretically the maximal one which can be directly extracted. The postprocessing technique we outlined leads to this rate.

3) Developoment and implementation of the postprocessing techniques in finite element programs is practically not a very simple task. We mention some aspects:

a) A number of extraction functions must be developed. Although many analytical solutions of special problems are very helpful for such development, the general approach especially for nonhomogeneous material still needs further research.

b) Special care must be excercised in the numerical evaluation of integrals because the extraction function can have singular character.

c) The postprocessing technique for nonlinear problems could be especially important but additional research is necessary.

REFERENCES

[1] Muskhelishvili, N. I., Some basic problems of the mathematical theory of elasticity (P. Noordhoff, Groningen, Netherlands, 1963).

[2] Basu, P. K., M. P. Rossow, B. S. Szabo, Theoretical

manual and user's guide for COMET-X (Center for Computational Mehanics, Washington University, St. Louis).

[3] Mesztenyi, C., W.Szymczak, FEARS user's manual for UNIVAC 1100 (University of Maryland, Institute for Physical Science and Technology Tech. Note BN-991, October 1982).

[4] Gignac, D. A., I. Babuška, C. Mesztenyi, An introduction to the FEARS program, David W. Taylor Naval Ship Research and Development Center Report DTNSRDC/CMLD-83/04, February 1983.

[5] Babuška, I., A. Miller, M. Vogelius, Adaptive methods and error estimation for elliptic problems of structural mechanics, University of Maryland, Institute for Physical Science and Technology, Tech. Note BN-1009, June 1983, to appear in the Proceedings of ARO Workshop on Adaptive Methods for Partial Differential Equations, SIAM, 1984.

[6] Babuška, I., M. Vogelius, Feedback and adaptive finite element solution in one-dimensional boundary value problems, University of Maryland, Institute for Physical Science and Technology Tech. Note 1006, October 1983.

[7] Babuška, I., W. C. Rheinboldt, Reliable error estimation and mesh adaptation for finite element method; in Computational methods in nonlinear mechanics (J. T. Oden, ed., North-Holland Publ. Co., Amsterdam, 1980, pp. 67-109).

[8] Babuška, I., B. A. Szabo,I. N. Katz, The p-version of the finite element method, SIAM, J. Numer. Anal 18 (1981) 515-545.

[9] Babuška, I., B. A. Szabo, On the rates of convergence of the finite element method, Internal J. Numer. Methods Engrg. 18 (1982) 323-341.

[10] Babuška, I., A. Miller, The post-processing in the finite element method, Part 1, Calculation of displacements stresses and other higher derivatives of displacements, University of Maryland, Institute for Physical Science and Technology, Tech. Note BN-992, December 1982. To appear in Internal J. Numer. Methods Engrg., 1984.

[11] Babuška, I., A. Miller, The post-processing approach in the finite element method, Part 2, The calculation of stress intensity factors, University of Maryland, Institute for Physical Science and Technology, Tech. Note BN 993, December 1982, to appear in Internal J. Numer. Methods Engrg., 1984.

[12] Babuška, I., A. Miller, The post-processing ap
the finite element method, Part 3, A posterior
estimates and adaptive mesh selection, Univers
Maryland, Institute for Physical Science and T
Tech. Note BN 1007, June 1983, to appear in In
Numer. Methods Engrg., 1984.

END

FILMED

6-84

DTIC